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Gauge Formulation of a Vector-Spinor Superalgebra

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We present a gauge formulation of the Poincaré algebra extended to include fermionic generators belonging to the vector-spinor representation of the Lorentz group. The gauge action of the theory, quadratic in curvature tensors, is derived.

Supersymmetric theories (Freedman *et al.*, 1976; Deser and Zumino, 1976; Grimm *et al.*, 1978; Mandelstam, 1983; Cremmer *et al.*, 1978) have been studied extensively as gauge theories which unify space-time and internal symmetries in a nontrivial way. In all these theories the fermionic generators of supersymmetry belong to the $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ representation of the Lorentz group. Theories of interest have one ($N = 1$), two ($N = 2$), four ($N = 4$ super Yang–Mills theory), and eight ($N = 8$ supergravity) such fermionic generators.

Supersymmetry algebra can be extended (Pilot and Rajpoot, 1989*a*,*b*) to include fermionic generators belonging to the vector-spinor representation of the Lorentz group. The resulting algebra circumvents the theorem of Haag, Lopuszanski, and Sohnius (1975) in a nontrivial way. As required by the theorem of Coleman and Mandula (1967), the $N = 1$ vector-spinor superalgebra closes on the generators of four-momentum P_a and the generators of rotations and boosts M_{ab} for a sensible S-matrix.

We take the generators of the $N = 1$ vector-spinor supersymmetry algebra to be P_a, M_{ab}, and $Q_{a\alpha}$. The nontrivial commutator and anticommutators of the algebra are $[M_{aa'}, M_{bb'}]$, $[M_{aa'}, P_b]$, $[M_{aa'}, Q_{b\beta}]$, $[M_{aa'}, \overline{Q}{}_b^\beta]$, and $\{Q_{a\alpha}, \overline{Q}{}_b^\beta\}$. It is convenient to write this algebra in the form $[X_A, X_B\} = if^C_{AB} X_C$,

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where X_A represents $\{M_{aa'}, P_a, Q_{a\alpha}\}$ and span a (6 + 4 + 12)-dimensional space. The structure constants of the algebra are

$$f^{cc'}_{aa'bb'} = -2[\eta_{ab}\delta^c_{a'}\delta^{c'}_{b'} - \eta_{ab'}\delta^c_{a'}\delta^{c'}_{b} - \eta_{a'b}\delta^c_{a}\delta^{c'}_{b'} + \eta_{a'b'}\delta^c_{a}\delta^{c'}_{b}] \tag{1}$$

$$f^{c}_{aa'b} = -(\eta_{ab}\delta^c_{a'} - \eta_{a'b}\delta^c_{a}) \tag{2}$$

$$f^{c\gamma}_{aa'b\beta} = -\tfrac{1}{2}\gamma^{\gamma}_{aa'\beta}\delta^c_b - (\eta_{ab}\delta^c_{a'}\delta^{\gamma}_{\beta} - \eta_{a'b}\delta^c_{a}\delta^{\gamma}_{\beta}) \tag{3}$$

$$f^{c}_{a\alpha b\beta} = -ia[\eta_{ab}(\gamma^c)^{\beta}_{\alpha} - \tfrac{1}{5}(\gamma_a)^{\beta}_{\alpha}\delta^c_b - \tfrac{1}{5}(\gamma_b)^{\beta}_{\alpha}\delta^c_a + \tfrac{3}{5}i\epsilon^c_{abd}(\gamma^d\gamma_5)^{\beta}_{\alpha}] \tag{4}$$

The structure constants satisfy the relation (Nath and Arnowitt, 1975; Mansouri, 1977) $f^C_{AB} = -(-1)^{\sigma(A)\sigma(B)}f^C_{BA}$, where $\sigma(A)$, $\sigma(B) = 1$ for fermionic generators and $\sigma(A)$, $\sigma(B) = 0$ for bosonic generators.

In order to construct a locally supersymmetric theory (MacDowell and Mansouri, 1977) with vector-spinor super-Poincaré algebra as the underlying local symmetry, we introduce space-time-dependent parameters $\lambda^A = (\lambda^{aa'}, \lambda^a, \lambda^{a\alpha})$ and gauge fields $A^A_\mu = (A^{aa'}_\mu, A^a_\mu, \bar{A}^{a\alpha}_\mu)$ for each generator $X_A = (M_{aa'}, P_a, Q_{a\alpha})$ of the algebra. Note that $A^{aa'}_\mu$ is the affine connection ($\omega^{aa'}_\mu$), A^a_μ is the vierbein (e^a_μ), and $\bar{A}^{a\alpha}_\mu = \bar{\psi}^{a\alpha}_\mu$ is an extension of the usual Rarita–Schwinger field $\bar{\psi}^{\alpha}_\mu$ with an additional vector index a. Next, we construct the following superalgebra-valued scalar and vector fields λ and A_μ:

$$\lambda = \lambda^A X_A = \tfrac{1}{2}\lambda^{aa'}M_{aa'} + \lambda^a P_a + \bar{\lambda}^{a\alpha}Q_{a\alpha} \tag{5}$$

$$A_\mu = A^A_\mu X_A = \tfrac{1}{2}\omega^{aa'}_\mu M_{aa'} + e^a_\mu P_a + \bar{\lambda}^{a\alpha}_\mu Q_{a\alpha} \tag{6}$$

The covariant derivative is defined as $D_\mu = \partial_\mu + iR_\phi(A_\mu)$, where $R_\phi(A_\mu)$ represents the appropriate action of gauge fields on the matter representation ϕ. Invariance of D_μ under translations requires $\delta(\lambda)(D_\mu\phi) = D_\mu(\delta(\lambda)\phi)$, where λ represents the parameters associated with the generators X_A. The gauge fields transform in the adjoint representation of the symmetry group, i.e., $A_\mu(x) = e^{-i\lambda}(A_\mu - i\partial_\mu)e^{i\lambda}$, and the infinitesimal version of this relation is

$$\partial A_\mu = \partial_\mu\lambda - i[\lambda, A_\mu] = D_\mu\lambda \tag{7}$$

Curvatures are defined through the use of the Ricci identity, $[D_\mu, D_\nu]\phi = iR_\phi(R_{\mu\nu})\phi$, where $R_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$. Thus the curvatures corresponding to the various generators of the algebra are $F^A_{\mu\nu} = (R^{aa'}_{\mu\nu}, F^a_{\mu\nu}, \bar{R}^{a\alpha}_{\mu\nu})$, where $R^{aa'}_{\mu\nu}$ represents the Riemann curvature tensor, $R^a_{\mu\nu}$ is the torsion, and $\bar{R}^{a\alpha}_{\mu\nu}$ is the SUSY (1, 1/2) + (1/2, 1) field strength associated with the extended Rarita–Schwinger field $\psi^{a\alpha}_\mu$. The superalgebra-valued curvature tensor $R_{\mu\nu}$ is defined as $R_{\mu\nu} = R^A_{\mu\nu}X_A = \tfrac{1}{2}R^{aa'}_{\mu\nu}M_{aa'} + R^a_{\mu\nu}P_a + \bar{R}^{a\alpha}_{\mu\nu}Q_{a\alpha}$ and transforms as $R_{\mu\nu} \to e^{-i\lambda}R_{\mu\nu}e^{i\lambda}$. Under infinitesimal gauge transformations we have $\delta R_{\mu\nu} = R^C_{\mu\nu}\lambda^B f^A_{BC}X_A$, a result which follows through the use of the Jacobi identity.

The curvature terms are

$$\begin{aligned} R_{\mu\nu}^{aa'} &= \partial_\mu \omega_\nu^{aa'} - \partial_\nu \omega_\mu^{aa'} + 2\omega_\mu^{ba}\omega_{\nu b}^{a'} \\ &\quad + e_\mu^a e_\nu^{a'} - e_\nu^a e_\mu^{a'} + \overline{\psi}_\mu^a \psi_\nu^{a'} - \overline{\psi}_\nu^a \psi_\mu^{a'} \\ &\quad + \tfrac{1}{2}\overline{\psi}_\mu^b \gamma^{aa'} \psi_{\nu b} \end{aligned} \tag{8}$$

$$\begin{aligned} R_{\mu\nu}^{a} &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu^{ba} e_{\nu b} - \omega_\nu^{ba} e_{\mu b} \\ &\quad + ia\overline{\psi}_\mu^b \gamma^a \psi_{\nu b} - \tfrac{3}{5} a\overline{\psi}_\mu^b \gamma^d \gamma_5 \psi_\nu^c \epsilon_{bcd}^a \end{aligned} \tag{9}$$

$$\begin{aligned} \overline{R}_{\mu\nu}^{a\alpha} &= \partial_\mu \overline{\psi}_\nu^{a\alpha} - \partial_\nu \overline{\psi}_\mu^{a\alpha} + \tfrac{1}{4}\omega_\mu^{bb'}(\overline{\psi}_\nu^a \gamma_{bb'})^\alpha \\ &\quad - \tfrac{1}{4}\omega_\nu^{bb'}(\overline{\psi}_\mu^a \gamma_{bb'})^\alpha + \omega_\mu^{ba}\overline{\psi}_{\nu\mu}^\alpha - \omega_\nu^{ba}\overline{\psi}_{\mu b}^\alpha \\ &\quad + iae_\mu^b(\overline{\psi}_\nu^a \gamma_b)^\alpha - iae_\nu^b(\overline{\psi}_\mu^a \gamma_b)^\alpha \\ &\quad - \frac{ia}{5} e_\mu^b(\overline{\psi}_{\nu b}\gamma^a)^\alpha + \frac{ia}{5} e_\nu^b(\overline{\psi}_{\mu b}\gamma^a)^\alpha \\ &\quad - \tfrac{3}{5} a[e_\mu^b(\overline{\psi}_\nu^c \gamma^d \gamma_s)^\alpha + e_\nu^b(\overline{\psi}_\mu^c \gamma^d \gamma_5)^\alpha]\epsilon_{bcd}^a \end{aligned} \tag{10}$$

In deriving these results, we have used the relation $f_{AB}^C = (-1) f_{BA}^C(-1)^{\sigma(C)[\sigma(A)+\sigma(B)]}$ and have taken $\psi_{\mu a}$ to be transverse, i.e., $(\gamma^a \psi_{a\mu})_\alpha = 0$. The Jacobi identities for the covariant derivatives D_λ imply the Bianchi identities $D_{(\lambda} R_{\mu\nu)}^A = 0$, where $(\lambda\mu\nu)$ stands for cyclic permutations of the indices.

An action quadratic in the curvature tensors is

$$I = \int d^4x\, e^{\mu\nu\sigma\tau} \zeta_{AB} R_{\mu\nu}^A R_{\sigma\tau}^B \tag{11}$$

where $e^{\mu\nu\sigma\tau} = e\epsilon^{\mu\nu\sigma\tau} = \sqrt{-g}\ \epsilon^{\nu\mu\sigma\tau}$ is a number and can be shown to be invariant due to the Schouten identity. Also, $e = \det(e_\mu^a)$, $g_{\mu\nu} = e_\mu^a(x) e_{\nu a}(x)$, $g = \det(g_{\mu\nu})$, and $\epsilon^{\mu\nu\sigma\tau}$ is the 4-dimensional Levi-Civita tensor and $\zeta_{AB} = \zeta_{BA}(-1)^{\sigma(A)\sigma(B)}$ is a constant metric used to contract the local, tangent space, or group indices. Invariance of the action under local gauge transformation requires the following condition to be satisfied (Pais and Rittenberg, 1975):

$$f_{AB}^D \zeta_{DC} = f_{CA}^D \zeta_{DB}(-1)^{\sigma(C)[\sigma(A)+\sigma(B)]}$$

This relation is satisfied provided we choose $\zeta_{AB} = (\eta_{ab}\eta_{a'b'} - \eta_{a'b}\eta_{ab'}, \eta_{ab}, \eta_{ab}\mathbf{1}_{\alpha\beta})$. Explicitly, the invariant action for $N = 1$ vector-spinor supergravity is

$$\begin{aligned} I = \int d^4x\, e^{\mu\nu\sigma\tau}[\tfrac{1}{2} R_{\mu\nu}^{aa'}(M) R_{\sigma\tau aa'}(M) \\ + R_{\mu\nu}^a(P) R_{a\sigma\tau}(P) + \overline{R}_{\mu\nu}^{a\alpha}(Q) R_{\sigma\tau a\alpha}(Q)] \end{aligned} \tag{12}$$

Inherent in the action are the usual terms that are topological invariants; terms that give Einstein's gravity with a cosmological constant and terms that are new and specific to the vector-spinor gauge theory. The action can be decomposed further into elements of irreducible representation of the gauge group. The decomposition is relevant for identifying the full symmetries of the action. The full implications of the theory with matter fields will be presented in a future publication.

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REFERENCES

Coleman, S., and Mandula, J. (1967). *Physical Review*, **159**, 1251.
Cremmer, E., Julia, B., and Scherk, J. (1978). *Physics Letters*, **76B**, 409.
Deser, S., and Zumino, B. (1976). *Physics Letters*, **62B**, 335.
Freedman, D. Z., van Nieuwenhuizen, P., and Ferrara, S. (1976). *Physical Review D*, **13**, 3214.
Grimm, R., Sohnius, M. F., and Wess, J. (1978). *Nuclear Physics B*, **133**, 275.
Haag, R., Lopuszanski, J., and Sohnius, M. (1975). *Nuclear Physics B*, **88**, 257.
MacDowell, S. W., and Mansouri, F. (1977). *Physical Review Letters*, **38**, 739.
Mandelstam, S. (1983). *Nuclear Physics B*, **213**, 149.
Mansouri, F. (1977). *Journal of Mathematical Physics*, **18**, 52.
Nath, P., and Arnowitt, R. (1975). *Physics Letters*, **56B**, 177.
Pais, A., and Rittenberg, V. (1975). *Journal of Mathematical Physics*, **16**, 2062.
Pilot, C., and Rajpoot, S. (1989*a*). *Modern Physics Letters A*, **3**, 303.
Pilot, C., and Rajpoot, S. (1989*b*). *Modern Physics Letters A*, **9**, 831.